Probability Theory

Textbook: A First Course in Probability, Sheldon Ross, 2019.

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Chapter 02: Axioms of Probability

Sample Space

A *sample space* is the set of all possible outcomes of an experiment whose outcome is not predictable with certainty.

Example 1: Sex of a newborn child

$$S = \{g, b\}$$

where the outcome g means that the child is a girl and b that it is a boy.

Example 2: Flipping two coins

$$S = \{(H,H),(H,T),(T,H),(T,T)\}$$

The outcome will be

- (H,H) if both coins are heads,
- (H,T) if the first coin is heads and the second tails,
- (T,H) if the first is tails and the second heads, and
- (T,T) if both coins are tails.

Sample Space (more examples)

Example 3: Order of finish in a horse race among 8 horses

$$S = \{\text{all } 8! \text{ permutations of } (1,2,\cdots,7,8)\}$$

Example 4: Tossing two dice

$$S = \{(i,j)\colon i,j = 1,2,\cdots,6\}$$

where the outcome (i,j) is said to occur if i appears on the leftmost die and j on the other die.

Example 5: Measuring (in hours) the lifetime of a capacitor

$$S = \{x\colon 0 \le x < \infty\}$$

Events

An *event* is any subset E of the sample space. An event is a set consisting of possible outcomes of the experiment. If the outcome of the experiment is contained in E, then we say that E has occurred.

Example 1: Event that the child is a girl

$$E = \{g\}$$

Example 2: Event that the two coins show different faces.

$$E = \{(H,T),(T,H)\}$$

Example 3: Event that horse 5 wins the race

$$E = \{\text{all outcomes in } S \text{ starting with a } 5\}$$

Example 5: Event that the capacitor lasts less than 200 hours

$$S = \{x\colon 0 \le x < 200\}$$

Union of Events

For any two events E and F of a sample space S, we define the new event $E \cup F$ to consist of all outcomes that are either in E or in F or in both E and F. That is, the event $E \cup F$ will occur if either E or F occurs.

Example 1: Sex of a newborn child

$E = \{g\}$ $\quad F = \{b\}$ $\quad E \cup F = \{g, b\} = S$

Example 2: Flipping two coins

$E = \{(H,T),(T,H)\}$ $\quad F = \{(T,T)\}$

$$E \cup F = \{(H,T),(T,H),(T,T)\}$$

$E \cup F$ will occur if a tail appeared on either coin.

$\bigcup_{i=1}^{\infty} E_i$ — If $E_1, E_2, \cdots$ are events, then the union of these events consists of all outcomes that are in E_i for at least one value of $i = 1, 2, \cdots$.

Intersection of Events

For any two events E and F of a sample space S, we define the new event EF (also written as $E \cap F$) to consist of all outcomes that are both in E and in F. That is, the event EF will occur only if both E and F occur.

Example 2: Flipping two coins

$E = \{(H,H),(H,T),(T,H)\}$	At least 1 head occurs
$F = \{((H,T),(T,H),(T,T)\}$	At least 1 tail occurs
$EF = \{(H,T),(T,H)\}$	Exactly 1 head and 1 tail occur

Example 4: Tossing two dice

$E = \{(1,4),(2,3),(3,2),(4,1)\}$		Sum of the dice is 5
$F = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$		Sum of the dice is 6
$EF = \varnothing$	(E and F are *mutually exclusive*)	Null event

$\bigcap_{i=1}^{\infty} E_i$ — If $E_1, E_2, \cdots$ are events, then the intersection of these events consists of those outcomes which are in all of these events.

Notation and Properties

Complement of an event E

- E^c consists of all outcomes in the sample space S that are not in E

Subset/Superset of an event

- $F \subset E$ (or $E \supset F$) if all outcomes in F are also in E
- If $F \subset E$ and $E \subset F \Rightarrow E = F$

From set theory

- Commutativity $E \cup F = F \cup E$ $\quad EF = FE$
- Associativity $(E \cup F) \cup G = E \cup (F \cup G)$ $\quad (EF)G = E(FG)$
- Distribution $(E \cup F)G = EG \cup FG$ $\quad EF \cup G = (E \cup G)(F \cup G)$
- DeMorgan's $\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$ $\quad \left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$

Definition of Probability

Probability as a long-run frequency of occurrence

- Consider an experiment whose sample space is S
- The experiment is repeated under exactly the same conditions
- $n(E)$: number of times, in the first n repetitions, that event E occurs
- $P(E)$: the probability of the event E, is defined as

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$$

- *Would $n(E)\big/n$ necessarily converge to some constant value?*

Probability as a measure of the degree of one's belief

- *Personal* or *subjective* view of probability.
- "There is a 30 percent chance of rain tomorrow"
- "It is 90 percent probable that Shakespeare actually wrote Hamlet"

Either interpretation yields the same mathematical properties

Axioms of Probability

- Consider an experiment whose sample space is S
- A number $P(E)$ is assumed for each event E of the sample space S
- $P(E)$ is defined and satisfies the following three axioms
- $P(E)$ is referred to as the probability of the event E.

Axioms of Probability

1. $0 \le P(E) \le 1$
2. $P(S) = 1$
3. For any sequence of *mutually exclusive* events $E_1, E_2, \cdots$,

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Direct Consequences

- $P(\varnothing) = 0$
- $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$

Examples of Probability

Example: Tossing a fair coin

A head is as likely to appear as a tail

$$P(\{H\}) = P(\{T\}) = 1/2$$

Example: Tossing a biased coin

If we felt that a head were twice as likely to appear as a tail

$$P(\{H\}) = 2/3 \qquad P(\{T\}) = 1/3$$

Example: Tossing a fair die

All six sides are equally likely to appear

$$P(\{1\}) = P(\{2\}) = P(\{3\}) = \cdots = P(\{6\}) = 1/6$$

$$\begin{aligned} \text{P(rolling an even number)} &= P(\{2,4,6\}) \\ &= P(\{2\}) + P(\{4\}) + P(\{6\}) = 1/2 \end{aligned}$$

Simple Propositions

$P(E^c) = 1 - P(E)$

$$\underbrace{1 = P(S)}_{\text{axiom 2}} = \overbrace{P(S) = P(E \cup E^c)}^{\text{complement}} \;;\; \underbrace{P(E \cup E^c) = P(E) + P(E^c)}_{\text{axiom 3}}$$

If $E \subset F$, then $P(E) \le P(F)$

$$\underbrace{P(F) = P(E \cup E^cF)}_{E\subset F \implies F = E\cup E^cF} = \overbrace{P(E) + P(E^cF)}^{\text{axiom 3: } E \text{ and } E^cF \text{ are disjoint}} \underbrace{\ge P(E)}_{\text{axiom 1: } P(E^cF)\ge 0}$$

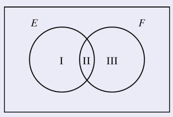
Simple Propositions (cont'd)

$P(E \cup F) = P(E) + P(F) - P(EF)$

$$\overbrace{P(E \cup F) = P(E \cup E^cF)}^{\text{rewriting as disjoint events}} = \underbrace{P(E) + P(E^cF)}_{\text{axiom 3}}$$

$$\overbrace{P(F) = P(EF \cup E^cF)}^{\text{rewriting as disjoint events}} = \underbrace{P(EF) + P(E^cF)}_{\text{axiom 3}}$$

Proof using Venn Diagram



$$\begin{aligned} E \cup F &= I \cup II \cup III & P(E \cup F) &= P(I) + P(II) + P(III) \\ E &= I \cup II & P(E) &= P(I) + P(II) \\ F &= II \cup III & P(F) &= P(II) + P(III) \\ EF &= II & P(EF) &= P(II) \end{aligned}$$

Simple Propositions (cont'd)

$P(E_1 \cup E_2 \cup E_3)$

$$
\begin{aligned}
P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) \\
- P(E_1E_2) - P(E_1E_3) - P(E_2E_3) \\
+ P(E_1E_2E_3)
\end{aligned}
$$

Inclusion-Exclusion Identity

$$
\begin{aligned}
P(E_1 \cup E_2 \cup \cdots E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \cdots \\
+ (-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1}E_{i_2}\cdots E_{i_r}) \\
+ \cdots + (-1)^{n+1} P(E_1E_2\cdots E_n)
\end{aligned}
$$

The summation $\sum_{i_1<i_2<\cdots<i_r} P(E_{i_1}E_{i_2}\cdots E_{i_r})$ is taken over all of the $\binom{n}{r}$ possible subsets of size r of the set $\{1, 2, \cdots, n\}$.

Examples

Example

J is taking two books along on her holiday vacation. With probability .5, she will like the first book; with probability .4, she will like the second book; and with probability .3, she will like both books. What is the probability that she likes neither book?

- Let B_i denote the event that J likes book i, $i = 1, 2$.
- The probability that she likes at least one of the books is

$$
\begin{aligned}
P(B_1 \cup B_2) &= P(B_1) + P(B_2) - P(B_1B_2) \\
&= .5 + .4 - .3 = .6
\end{aligned}
$$

- The probability that she likes neither books is

$$
\begin{aligned}
P(B_1^cB_2^c) &= P((B_1 \cup B_2)^c) \\
&= 1 - P(B_1 \cup B_2) \\
&= .4
\end{aligned}
$$

Examples (cont'd)

The matching problem

Suppose that each of N men at a party throws his hat into the center of the room. The hats are first mixed up, and then each man randomly selects a hat. How likely is it that none of the men selects his own hat?

- Let E_i denote the event that the i^{th} man selects his own hat.
- The probability that at least one of the men selects his own hat

$$
\begin{aligned}
P\left(\bigcup_{i=1}^{N} E_i\right) &= \sum_{r=1}^{N} (-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1}E_{i_2}\cdots E_{i_r}) \\
&= \sum_{r=1}^{N} (-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} \frac{(N-r)!}{N!} \\
&= \sum_{r=1}^{N} (-1)^{r+1} \binom{N}{r} \frac{(N-r)!}{N!} = \sum_{r=1}^{N} (-1)^{r+1} \frac{1}{r!}
\end{aligned}
$$

Examples (cont'd)

The matching problem (cont'd)

Suppose that each of N men at a party throws his hat into the center of the room. The hats are first mixed up, and then each man randomly selects a hat. How likely is it that none of the men selects his own hat?

- The probability that none of the men selects his own hat

$$
\begin{aligned}
P\left(\left(\bigcup_{i=1}^{N} E_i\right)^c\right) &= 1 - P\left(\bigcup_{i=1}^{N} E_i\right) \\
&= 1 - \sum_{r=1}^{N} (-1)^{r+1} \frac{1}{r!} \\
&= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots + (-1)^N \frac{1}{N!} \\
\lim_{N\to\infty} P\left(\ \text{"}\ \right) &= e^{-1} \approx .36788
\end{aligned}
$$

Sample Spaces Having Equally Likely Outcomes

Consider an experiment whose sample space S is a finite set, say, $S = \{1, 2, \cdots, N\}$. Then it is often natural to assume that

$$P(\{1\}) = P(\{2\}) = \cdots = P(\{N\})$$

which implies, from Axioms 2 and 3, that

$$P(\{i\}) = \frac{1}{N} \qquad i = 1, 2, \cdots, N$$

From Axiom 3

The probability of any event E equals the proportion of outcomes in the sample space that are contained in E.

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}$$

Examples

Example 1

Two fair dice are rolled at random. How likely is it that both show 6?

Ordered with Replacement

$$
\begin{aligned}
N(S) &= 6^2 = 36 \\
N(E) &= 1
\end{aligned}
$$

$$P(E) = \frac{N(E)}{N(S)} = \frac{1}{36}$$

Unordered with Replacement

$$N(S) = \binom{6+2-1}{2} = 21$$

$$P(x, y) = \begin{cases} q, & \text{if } x = y \\ 2q, & \text{if } x \neq y \end{cases}$$

$$
\begin{aligned}
1 &= P(S) \\
&= 6 \times q + (21 - 6) \times 2q
\end{aligned}
$$

$$P(E) = q = \frac{1}{36}$$

Examples (cont'd)

Example 2

If 3 balls are "randomly drawn" from a bowl containing 8 white and 4 black balls, what is the probability that one of the balls is black and the other two white?

$$\frac{\binom{4}{1}\binom{8}{2}}{\binom{12}{3}} = .51$$

Example 3: The birthday paradox

If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$?

$$\frac{{}^{365}P_n}{365^n}$$

$$n \leq 23$$

Example 4

In the game of bridge, the entire deck of 52 cards is dealt out to 4 players. What is the probability that each player receives 1 ace?

$$\frac{4!\binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}} = .11$$

Countably-Infinite Sample Spaces

Consider and event E of a countably-infinite sample space S.

$$S = \bigcup_{i=1}^{\infty} \{w_i\} \qquad E = \bigcup_{i=1}^{\infty} \Big(\{w_i\} \cap E\Big)$$

From Axiom 3

$$P(E) = \sum_{i=1}^{\infty} P\Big(\{w_i\} \cap E\Big) = \sum_{\substack{i=1 \\ w_i \in E}}^{\infty} P(w_i)$$

Try to find an analytical form for $P(w_i)$ for all $w_i \in E$.

Example

Example

A fair die is being thrown repeatedly until a 6 appears. Let X be the number of tosses until 6 appears. Find $P(A)$ and $P(B)$, where:

- $A \triangleq X$ is even
- $B \triangleq X$ is odd

$$P(X=1) = \frac{1}{6}$$

$$P(X=2) = \frac{5}{6^2}$$

$$P(X=3) = \frac{5^2}{6^3}$$

$$P(X=4) = \frac{5^3}{6^4}$$

$$P(X=x) = \frac{1}{6}\left(\frac{5}{6}\right)^{x-1}$$

$$P(A) = P(2) + P(4) + P(6) + \dots = \frac{1}{6} \times \left[\frac{5}{6} + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^5 + \dots\right] = \frac{1}{6} \times \frac{5/6}{1-(5/6)^2} = \frac{5}{11}$$

$$P(B) = 1 - P(A) = \frac{6}{11}$$